SPECIAL VALUES OF ANTICYCLOTOMIC L-FUNCTIONS MODULO λ

ALIA HAMIEH

ABSTRACT. The purpose of this article is to generalize some results of Vatsal on studying the special values of Rankin-Selberg L-functions in an anticyclotomic $\mathbb{Z}_p$-extension. Let g be a cuspidal Hilbert modular form of parallel weight $(2, ..., 2)$ and level $\mathcal{N}$ over a totally real field F, and let K/F be a totally imaginary quadratic extension of relative discriminant $\mathcal{D}$. We study the l-adic valuation of the special values $L(g, \chi, \frac{1}{2})$ as χ varies over the ring class characters of K of $\mathcal{P}$-power conductor, for some fixed prime ideal $\mathcal{P}$. We prove our results under the only assumption that the prime to $\mathcal{P}$ part of $\mathcal{N}$ is relatively prime to $\mathcal{D}$.

0. INTRODUCTION

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N, and let $K/\mathbb{Q}$ be an imaginary quadratic field extension of discriminant D such that N and D are relatively prime. Denote by K_∞ the anticyclotomic $\mathbb{Z}_p$-extension of K where p is a given prime number with $p \nmid ND$. In 2002, Vatsal succeeded in settling a conjecture of Mazur pertaining to the size of the Mordell-Weil group $E(K_\infty)$. In fact, Mazur's conjecture predicts that the group $E(K_\infty)$ is finitely generated, and Vatsal proved in [6] that this is true, at least when E is ordinary at p, or when the class number of K is prime to p.

In more concrete terms, Vatsal considered the modular form g associated to E and the family of Rankin -Selberg L-functions $L(g, \chi, s)$ as χ varies over ring class characters of K of p-power conductor. Under certain conditions on g and χ, the result of Vatsal asserts that the special values $L(g, \chi, 1)$ are non-vanishing for all but finitely many χ, provided that p is an ordinary prime for g or p does not divide the class number of K. One consequence of this result is the non-triviality of certain Euler systems as formulated by Bertolini-Darmon in [1] which in its turn implies that the desired statement about the Mordell-Weil group is indeed true.

In 2004, Cornut and Vatsal generalized in [3] the above mentioned work of Vatsal to totally real fields. Numerous technical complications arise due to the fact that a more general number field F is considered. However, the basic arguments are ultimately the same, as the authors invoke deep theorems of Ratner [5] on uniform distribution of unipotent orbits on p-adic Lie groups to deduce the desired result.

In 2003, Vatsal extended the results and methods of [6] to study the variation of the λ-adic absolute value of $L^{\mathrm{al}}(g, \chi, 1)$ as a function of χ, where λ is a fixed prime of $\bar{\mathbb{Q}}$ with

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, ROOM 121, 1984 MATHEMATICS ROAD, VANCOUVER, BRITISH COLUMBIA, CANADA V6T 1Z2

E-mail address: ahamieh@math.ubc.ca.

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residue characteristic l. The object of this paper is to generalize this work to totally real fields while removing most of the restrictions on N, p, D and l (Theorem 4.14). We use the improved formalism developed in [3] to achieve this purpose.

We now give a brief account of the results in this work. Let π be an irreducible automorphic representation of GL_2 over a totally real field corresponding to a cuspidal Hilbert modular newform g of level $\mathcal{N}$, trivial character and parallel weight $(2, ..., 2)$. The Hecke eigenvalues of g are denoted by a_v ($T_v g = a_v g$). Let $\mathcal{P}$ be a prime ideal of F such that $\mathcal{P}$ lies over an odd rational prime p. Let l be a rational prime, and denote by E_l the l-adically complete discrete valuation ring containing the Hecke eigenvalues of g. To simplify the exposition of the introduction, we assume that $l \neq p$ although we should mention that the case $l = p$ does not give rise to significant complications. Let χ be a ring class character of K of conductor $\mathcal{P}^n$ such that $\chi = 1$ when restricted to $\mathbb{A}_F^* \subset \mathbb{A}_K^*$. In addition to some mild restrictions, we assume that the prime to $\mathcal{P}$ part of $\mathcal{N}$ is relatively prime to the discriminant $\mathcal{D}$ of K/F. We also impose sufficient conditions to make the sign in the functional equation of $L(\pi, \chi, s)$ equal $+1$ for all but finitely many characters χ of the type considered above.

Let $G(n)$ be the Galois group of the ring class field of conductor $\mathcal{P}^n$ over K. We have the decomposition $G(n) = G_0 \times H(n)$ where G_0 is known as the tame subgroup of G(n) and $H(n)$ as the wild subgroup of $G(n)$. By class field theory, we can view χ as a character of $G(n)$. Hence, χ can be written as the product of a tamely ramified character χ_0 of G_0 and a wild character χ_1 of $H(n)$. It can be shown that $G(n)$ acts simply transitively on the set of CM points of conductor $\mathcal{P}^n$ on the Shimura curve associated to some carefully chosen totally definite quaternion algebra B. Given a CM point x of conductor $\mathcal{P}^n$ and a ring class character χ of the same conductor, we define the Gross-Zagier sum

$$\mathrm{a}(x, \chi) = \frac{1}{|G(n)|} \sum_{\sigma \in G(n)} \chi(\sigma)\psi(\sigma.x),$$

where ψ is the E_l-valued function on the set of CM points, associated to g via the Jacquet-Langlands correspondence. In the light of the existing Gross-Zagier formulae, our job is reduced to studying the l-adic valuation of this sum. More precisely, our goal is to prove an analogue of Proposition 4.1 and Corollary 4.2 in [7] for a Hilbert modular form g over a totally real field F, while removing the assumptions on l and the order of χ_0. Before we state the results we obtained in this direction, it is perhaps more enlightening to shed some light on Vatsal's result in which π corresponds to a weight 2 newform g for $\Gamma_0(N)$ such that N, p, and D are pairwise relatively prime.

Theorem 0.1. *Let the tame character χ_0 be given such that its order is prime to p. Then, under some restrictions on l and the Hecke field of g, we have*

$$ord_\lambda(\mathrm{a}(x, \chi)) < \mu \tag{1}$$

for all $n \gg 0$ and $\chi = \chi_0\chi_1$, where μ is the smallest integer such that $a_q \not\equiv 1 + q \mod \lambda^\mu$ for some $q \nmid pND$.

Note that the restriction on the Hecke field of g, which will be made clear in Section 4, was overlooked in [7]. More importantly, we remark that (1) is mistakenly given as an

equality in [7] due to an error made in the proof of Proposition 5.3 part (2). We mention here that in order to make the necessary corrections, we modify the definition of the constant μ from the one given in [7]. We say a bit more about these issues in Section 4.

Recall that E_l is an l-adically complete discrete valuation ring containing the Hecke eigenvalues of g. We may assume without loss of generality that E_l contains the values of χ_0 and the p-th roots of unity. We consider the trace of $\mathrm{a}(x,\chi)$ taken from $E_l(\chi_1)$ to E_l:

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \sum_{\sigma \in \mathrm{Gal}(E_l(\chi_1)/E_l)} \sigma(\mathrm{a}(x,\chi)).$$

This trace expression is different than the average expression

$$\mathrm{b}(x,\chi_0) = \sum_{\chi_1 \in \widehat{H(n)}} \mathrm{a}(x,\chi_0\chi_1)$$

considered in the work of Vatsal and Cornut-Vatsal. In particular, given any $\chi = \chi_0\chi_1$, the non-vanishing of $\mathrm{Tr}(\mathrm{a}(x,\chi))$ implies that of $\mathrm{a}(x,\chi)$. After a series of reductions, we prove the following proposition in two level raising steps (Proposition 4.9 and Proposition 4.12).

Proposition 0.2. *The trace expression simplifies to*

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \frac{|G_2|[E_l(\chi_1):E_l]}{|G(n)|} \sum_{\sigma \in G_0/G_1} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.x_{m,\mathcal{D}}),$$

where G_1 and G_2 are certain subgroups of $G(n)$, $\psi_{m,\mathcal{D}}$ is a function of higher level induced by ψ, and $x_{m,\mathcal{D}}$ is a CM point of higher level and conductor $\mathcal{P}^n$.

Hence, the problem is reduced to studying the λ-adic valuation of

$$\sum_{\sigma \in G_0/G_1} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.x_{m,\mathcal{D}}).$$

Finally, we arrive at the following theorem (Theorem 4.14 in Section 4).

Theorem 0.3. *Let χ_0 be any character of G_0. For any CM point x of conductor $\mathcal{P}^n$ with $n \gg 0$, there exists some $y \in G(n).x$ such that*

$$ord_\lambda\left(\sum_{\sigma \in G_0/G_1} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.y)\right) < \mu,$$

where μ is precisely given in Section 4 Definition 4.8.

The organization of this paper is as follows. In Section 1, we introduce some notation and fix a set of hypotheses which we require throughout the chapter. In Section 2, we recall the construction of CM points associated to a totally definite quaternion algebra. In Section 3, we recall the fundamental result of Cornut-Vatsal [3] on uniform distribution of CM points. In Section 4, we explore the Gross-Zagier sum $\mathrm{a}(x,\chi)$ which is closely related to the special value $L(\pi,\chi,\frac{1}{2})$. Then we prove some propositions and lemmas related to this sum leading to the main result Theorem 4.14.

1. Preliminaries and Notations

Let us first fix some notation. We write $\mathbb{A}$ (resp. $\mathbb{A}_f$) for the ring of adeles (resp. finite adeles) of $\mathbb{Q}$. Let F be a totally real number field, and let K be a totally imaginary quadratic extension of F the discriminant of which we denote by D. The ring of adeles of F is $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$, and the ring of finite adeles of F is $\hat{F} = \mathbb{A}_f \otimes_{\mathbb{Q}} F$. Similarly, we write $\mathbb{A}_K$ (resp. $\hat{K}$) for the ring of adeles (resp. finite adeles) of K.

We consider a cuspidal Hilbert modular newform g of level $\mathcal{N}$, trivial character and parallel weight $(2, ..., 2)$. We denote by a_v the Hecke eigenvalues of g ($T_v g = a_v g$) and by π the automorphic irreducible representation of GL_2 over F corresponding to g. Let $\mathcal{P}$ be a prime ideal of F such that $\mathcal{P}$ lies over an odd rational prime p, and let $\pi_{\mathcal{P}}$ be a uniformizer of $F_{\mathcal{P}}$. By abuse of notation, we also denote the maximal ideal in $O_{F,\mathcal{P}}$ by $\mathcal{P}$. Let χ be a finite-order Hecke character of K of conductor $\mathcal{P}^n$.

The data of the previous paragraph is to remain fixed and the following hypotheses are assumed throughout this article:

(1) The representations π and $\pi \otimes \eta$ are distinct, where η is the quadratic character associated to the extension K/F. We say that the pair (π, K) is non-exceptional.

(2) The prime to $\mathcal{P}$ part $\mathcal{N}'$ of $\mathcal{N}$ is relatively prime to the discriminant D of K/F

(3) The character χ is a ring class character of $\mathcal{P}$-power conductor, and it is trivial when restricted to $\mathbb{A}_F^* \subset \mathbb{A}_K^*$.

(4) Let S be the set of all the Archimedean places of F, together with those finite places of F which do not divide $\mathcal{P}$, are inert in K, and divide $\mathcal{N}$ to an odd power. We require S to have an even cardinality

It follows from the last condition that the sign in the functional equation of $L(\pi, \chi, s)$ is $+1$ for all but finitely many characters χ that satisfy condition (2) (see Lemma 1.1 in [3]).

Let l be any rational prime. Fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_l}$, and denote by E the subalgebra of $\overline{\mathbb{Q}_l}$ generated by the images of the Hecke eigenvalues of g. Write E_l for the integral closure of E in its field of fractions and λ for the maximal ideal in E_l.

2. CM Points and Galois Action

Let B be the totally definite quaternion algebra over F such that $\mathrm{Ram}(B) = S$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(B^*)$ be the algebraic group over $\mathbb{Q}$ associated to B^*. Thus, the center of G is $Z = \mathrm{Res}_{F/\mathbb{Q}}(F^*)$. Since every place in F that ramifies in B is inert in K, there exists an F-embedding $K \hookrightarrow B$. After fixing such an embedding, the group $T = \mathrm{Res}_{F/\mathbb{Q}}(K^*)$ can be viewed as a maximal sub-torus of G defined over $\mathbb{Q}$.

In what follows, we sketch the construction of an O_F-order R of reduced discriminant $\mathcal{N}$ in B following [3] and [11]. Let $\mathcal{N}'$ be the prime to $\mathcal{P}$ part of $\mathcal{N}$, and write $\mathcal{N} = \mathcal{P}^\delta \mathcal{N}'$. Let R_0 be an Eichler order of level $\mathcal{P}^\delta$ in B. We choose R_0 such that the O_F-order $O = O_K \cap R_0$ has a $\mathcal{P}$-power conductor. For example, if $\mathcal{P}$ does not divide $\mathcal{N}$, we require that R_0 optimally contains O_K. Denote by $\mathcal{N}_B$ the discriminant of B/F, and let $\mathcal{M}_K$ be an ideal in O_K which

has relative norm $\mathcal{N}'/\mathcal{N}_B$. We may find such an ideal $\mathcal{M}_K$ as follows. For each prime $\mathfrak{P}$ dividing $\mathcal{N}'$, let $\mathfrak{P}_K$ be a prime of O_K dividing $\mathfrak{P}$. If we put

$$\mathcal{M} = \prod_{\mathfrak{P}|\mathcal{N}_B} \mathfrak{P}_K^{[ord_{\mathfrak{P}}(\mathcal{N})/2]} . \prod_{\mathfrak{P}|\frac{\mathcal{N}'}{\mathcal{N}_B}} \mathfrak{P}_K^{ord_{\mathfrak{P}}(\mathcal{N})},$$

then

$$\mathcal{M}_K = \prod_{\mathfrak{P}} \mathfrak{P}_K^{ord_{\mathfrak{P}}(\mathcal{M})}.$$

Finally, we obtain R by the following formula:

$$R = O + (O \cap \mathcal{M}_K).R_0.$$

In particular, $R_{\mathcal{P}} = R_{0,\mathcal{P}}$ is an Eichler order of level $\mathcal{P}^\delta$. Define an open compact subgroup H of $G(\mathbb{A}_f)$ by $H = \hat{R}^*$. The subgroup H is sometimes referred to as the level structure. This gives rise to the finite sets

$$\mathrm{M}_H = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/H,$$

and

$$\mathrm{N}_H = Z(\mathbb{Q})^+\backslash Z(\mathbb{A}_f)/\mathrm{nrd}(H).$$

It also gives rise to the set of CM points

$$\mathrm{CM}_H = T(\mathbb{Q})\backslash G(\mathbb{A}_f)/H.$$

Notice that any function on M_H induces a function on CM_H via the obvious reduction map red : $CM_H \to \mathrm{M}_H$.

The action of $T(\mathbb{A}_f)$ on CM_H by left multiplication in $G(\mathbb{A}_f)$ factors through the reciprocity map $\mathrm{rec}_K : T(\mathbb{A}_f) \to \mathrm{Gal}_K^{\mathrm{ab}}$. This induces an action of $\mathrm{Gal}_K^{\mathrm{ab}}$ on CM_H. Hence, for $x = [g] \in \mathrm{CM}_H$ and $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$, we have $\sigma.x = [\beta g]$ where $\beta \in T(\mathbb{A}_f)$ is such that $\mathrm{rec}_K(\beta) = \sigma$.

Moreover, the reduced norm map on $G(\mathbb{A}_f)$ induces the map c : $\mathrm{M}_H \to \mathrm{N}_H$. Hence, the action of $\mathrm{Gal}_F^{\mathrm{ab}}$ on N_H induces an action of $\mathrm{Gal}_K^{\mathrm{ab}}$ on N_H. For $x = [z] \in \mathrm{N}_H$ and $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$, we have $\sigma.x = [\mathrm{nrd}(\beta)g]$ where $\beta \in T(\mathbb{A}_f)$ is such that $\mathrm{rec}_K(\beta) = \sigma$.

We now introduce the notion of a CM point with a $\mathcal{P}$-power conductor.

Definition 2.1. *We say that $x = [g]$ is a CM point of conductor $\mathcal{P}^n$ and write $x \in \mathrm{CM}_H(\mathcal{P}^n)$ if $T(\mathbb{A}_f) \cap gHg^{-1} = \widehat{O_{\mathcal{P}^n}}^*$, where $O_{\mathcal{P}^n} \subset O_K$ is the O_F-order of conductor $\mathcal{P}^n$.*

Choose $\alpha_{\mathcal{P}} \in O_{K,\mathcal{P}}$ such that $\{1, \alpha_{\mathcal{P}}\}$ is an $O_{F,\mathcal{P}}$-basis of $O_{K,\mathcal{P}}$. Since $O_{\mathcal{P}^n,\mathcal{P}} = O_{F,\mathcal{P}} + \mathcal{P}^n O_{K,\mathcal{P}}$, the set $\{1, \pi_{\mathcal{P}}^n \alpha_{\mathcal{P}}\}$ is an $O_{F,\mathcal{P}}$-basis of $O_{\mathcal{P}^n,\mathcal{P}}$. We fix the embedding $K_{\mathcal{P}} \hookrightarrow M_2(F_{\mathcal{P}})$ defined by

$$a + b\alpha_{\mathcal{P}} \mapsto \begin{pmatrix} a + b\mathrm{Tr}\alpha_{\mathcal{P}} & b\mathrm{N}\alpha_{\mathcal{P}} \\ -b & a \end{pmatrix},$$

where Tr and N denote the trace and norm maps.

Lemma 2.2. *Consider $g_{\mathcal{P}} \in B_{\mathcal{P}} \simeq M_2(F_{\mathcal{P}})$ specified as:*

$$g_{\mathcal{P}} = \begin{pmatrix} \pi_{\mathcal{P}}^n \mathrm{N}\alpha_{\mathcal{P}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for n large enough, the order $g_{\mathcal{P}} R_{\mathcal{P}} g_{\mathcal{P}}^{-1}$ in $B_{\mathcal{P}}$ optimally contains the $O_{F,\mathcal{P}}$-order in $K_{\mathcal{P}}$ of conductor $\mathcal{P}^n$.

Proof. Let $\tau = a + b\alpha_{\mathcal{P}}$ be any element in $K_{\mathcal{P}}$. We have

$$g_{\mathcal{P}}^{-1} \tau g_{\mathcal{P}} = \begin{pmatrix} a + b\mathrm{Tr}\alpha_{\mathcal{P}} & b\pi_{\mathcal{P}}^{-n} \\ -b\pi_{\mathcal{P}}^n \mathrm{N}\alpha_{\mathcal{P}} & a \end{pmatrix}.$$

Recall that, by construction, $R_{\mathcal{P}}$ is an Eichler order (of level $\mathcal{P}^\delta$) in $B_{\mathcal{P}} \simeq M_2(F_{\mathcal{P}})$. Without loss of generality, we identify $R_{\mathcal{P}}$ with the order

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(O_{F,\mathcal{P}}) : c \equiv 0 \mod \pi_{\mathcal{P}}^\delta \right\}.$$

Then, for all $n \gg 0$, we have $g_{\mathcal{P}}^{-1} \tau g_{\mathcal{P}} \in R_P$ if and only if $\tau \in O_{\mathcal{P}^n,\mathcal{P}}$. In other words, the order $g_{\mathcal{P}} R_{\mathcal{P}} g_{\mathcal{P}}^{-1}$ in $B_{\mathcal{P}}$ optimally contains the $O_{F,\mathcal{P}}$-order in $K_{\mathcal{P}}$ of conductor $\mathcal{P}^n$. □

In the sequel, we shall fix a choice of CM point $x = [g] \in CM_H(\mathcal{P}^n)$ such that it's $\mathcal{P}$-*th* component is as specified in the previous lemma.

For later reference, notice that $R_{\mathcal{P}}^*$ is isomorphic to the Iwahori subgroup

$$U_0(O_{F,\mathcal{P}}, \mathcal{P}^\delta) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(O_{F,\mathcal{P}}) : c \equiv 0 \mod \pi_{\mathcal{P}}^\delta \right\}.$$

3. Uniform Distribution of CM Points

The CM points are uniformly distributed on the components of the Shimura curve associated to B. This was the most crucial idea behind Vatsal's proof of Mazur's conjecture for weight two modular forms over $\mathbb{Q}$. In this section, we recall a crucial result on the uniform distribution of CM points due to Cornut-Vatsal, which we use to prove our main theorem. To describe this result, we need to introduce some more notation.

Let $K[\mathcal{P}^n]$ be the ring class field over K of conductor $\mathcal{P}^n$. In other words, $K[\mathcal{P}^n]$ is the abelian extension of K associated by class field theory to the subgroup $K^* K_\infty^* \hat{O}_{\mathcal{P}^n}^*$ of $\mathbb{A}_K^*$. Let $G(n)$ denote the Galois group of this extension. We have

$$G(n) = \mathrm{Gal}(K[\mathcal{P}^n]/K) \simeq \mathbb{A}_K^* / (K^* K_\infty^* \hat{O}_{\mathcal{P}^n}^*)$$

via the reciprocity map of K.

Set $K[\mathcal{P}^\infty] = \cup_{n \geq 0} K[\mathcal{P}^n]$, so that $G(\infty) = \mathrm{Gal}(K[\mathcal{P}^\infty]/K)$. The torsion subgroup of $G(\infty)$ is denoted by G_0. It is finite and $G(\infty)/G_0$ is a free $\mathbb{Z}_p$-module of rank $[F_{\mathcal{P}} : \mathbb{Q}_p]$. The reciprocity map of K maps $\mathbb{A}_F^* \subset \mathbb{A}_K^*$ onto the subgroup $G_2 \simeq Pic(O_F)$ of G_0.

Let $G(\infty)'$ be the subgroup of $G(\infty)$ generated by the Frobeniuses of the primes of K which are not above $\mathcal{P}$. Write $G_1 = G_0 \cap G(\infty)'$. Let $\mathcal{D}'$ be the square-free product of the primes $\mathcal{Q} \neq \mathcal{P}$ of F which ramify in K. Then G_1/G_2 is an $\mathbb{F}_2$-vector space with basis

$$\{\sigma_{\mathcal{Q}} \mod G_2 : \mathcal{Q}|\mathcal{D}'\},$$

where $\sigma_{\mathcal{Q}} = \mathrm{Frob}_{\mathfrak{Q}}$ and $\mathfrak{Q}$ is the prime of K above $\mathcal{Q}$.

Loosely speaking, the uniform distribution result in [6] states the following. Let p_1 and p_2 be arbitrary double cosets in M_H, and let σ be an arbitrary nontrivial element of G_0 with $\sigma \notin G_1$. Then there exists a CM point $x \in CM(\mathcal{P}^n)$ such that $\mathrm{red}(x) = p_1$ and $\mathrm{red}(\sigma.x) = p_2$ whenever n is sufficiently large.

In what follows, we describe the result of Cornut and Vatsal which extends and refines Vatsal's theorem alluded to in the previous paragraph.

Let $\mathcal{R}$ be a set of representatives for G_0/G_1 containing 1. We have the following maps:

$$\mathrm{RED} : CM_H(\mathcal{P}^\infty) \to M_H^{\mathcal{R}}, \quad x \mapsto (\mathrm{red}(\tau.x))_{\tau\in\mathcal{R}}$$

$$\mathrm{C} : M_H^{\mathcal{R}} \to N_H^{\mathcal{R}}, \quad (a_\tau)_{\tau\in\mathcal{R}} \mapsto (c(a_\tau))_{\tau\in\mathcal{R}}$$

and the composite map

$$\mathrm{C} \circ \mathrm{RED} : CM_H(\mathcal{P}^\infty) \to N_H^{\mathcal{R}},$$

which is $G(\infty)$-equivariant.

The following is the key theorem of Cornut-Vatsal as stated in [3]. However, the reader is referred to [2] for a proof of this result.

Theorem 3.1 (See [3]). *For all but finitely many $x \in CM_H(\mathcal{P}^\infty)$,*

$$\mathrm{RED}(G(\infty).x) = \mathrm{C}^{-1}(G(\infty).C \circ \mathrm{RED}(x))$$

4. Toward Computing $ord_\lambda(L^{\mathrm{al}}(\pi, \chi, \frac{1}{2}))$

Let π' be the unique cuspidal automorphic representation on B that is associated to π by the Jacquet-Langlands correspondence ($\pi = \mathrm{JL}(\pi')$). We associate to g a unique function $\theta \in S_2(\pi')$, where $S_2(\pi')$ is the representation space of π'. One can view θ as

$$\theta : \mathrm{M}_H \to E_g,$$

where E_g is the Hecke field of g. This yields the function $\psi = \theta \circ \mathrm{red}$ on CM_H. The space of functions on M_H is endowed by an action of Hecke operators T_v. This action agrees with the classical Hecke action on the space of Hilbert modular forms in the sense that θ has the same eigenvalues as g for all T_v, $v \nmid \mathcal{N}$. In particular, we may view θ as taking values in E_l:

$$\theta : \mathrm{M}_H \to E_l.$$

Without loss of generality, we may also assume that $\theta([g])$ is a λ-adic unit for some $[g] \in \mathrm{M}_H$.

There are several results in the literature which relate the special value $L(\pi, \chi, \frac{1}{2})$ to $|\mathrm{a}(x, \chi)|^2$, for some CM point $x \in CM_H(\mathcal{P}^n)$, with

$$\mathrm{a}(x, \chi) = \frac{1}{|G(n)|} \sum_{\sigma \in G(n)} \chi(\sigma)\psi(\sigma.x).$$

This kind of relations has been the subject of extensive research for over 25 years now. In 1985, Waldspurger proved a fundamental theorem (Théorème 2 in [9]) yielding a criterion for the non-vanishing of $L(\pi, \chi, \frac{1}{2})$ in a very general setting. Roughly speaking, the result of Waldspurger states that, under very mild conditions on π and χ, $L(\pi, \chi, \frac{1}{2}) \neq 0$ if and only if $|\mathrm{a}(x, \chi)|^2 \neq 0$. However, this result doesn't give a precise formula for the special value $L(\pi, \chi, \frac{1}{2})$ in terms of $|\mathrm{a}(x, \chi)|^2$. Most authors refer to such a formula as a Gross-Zagier formula, and it is expected (but not known yet) that it exists in full generality. Nevertheless, Zhang has proven a formula of this type in [10] under the assumption that the central character of π is trivial and that $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{D}$ are pairwise co-prime. Another significant improvement in this direction has been made in [4] by Martin and Whitehouse who established a Gross-Zagier formula under the only assumption that $\mathcal{P}$ does not divide $\mathcal{N}$.

Results as such were the point of departure in the work of Vatsal [6] and Cornut-Vatsal [3] on the non-vanishing of $L(\pi, \chi, \frac{1}{2})$. However, in order to study this special value modulo a given prime $\lambda \in \bar{\mathbb{Q}}$, Vatsal employed in [7] a construction of Hida to define a canonical period Ω_g^{can} such that

$$L^{\mathrm{al}}(\pi, \chi, \frac{1}{2}) = \frac{L(\pi, \chi, \frac{1}{2})}{\Omega_g^{\mathrm{can}}}$$

is a λ-adic integer. Then Vatsal incorporated this period into the Gross-Zagier formula obtained by Zhang to deduce in one simple step an exact statement about the λ-adic valuation of $L(\pi, \chi, \frac{1}{2})$ from that of $\mathrm{a}(x, \chi)$. Unfortunately, such a construction is not yet known in the degree of generality required in this paper. Hence, we will only concern ourselves with studying the value of $\mathrm{a}(x, \chi)$ modulo a given prime ideal $\lambda \subset \bar{\mathbb{Q}}$ as χ varies over the ring class characters of K of $\mathcal{P}$-power conductor.

Adapting the notation from [3], we identify ring class characters of $\mathcal{P}$-power conductor with finite-order characters of $G(\infty)$. Hence, given a character χ_0 of G_0 such that $\chi_0 = 1$ on G_2, denote by $P(n, \chi_0)$ the set of primitive characters of $G(n)$ (do not factor through $G(n-1)$) and induce χ_0 on G_0. In [6] and [3], the authors proved that for each character χ_0 of G_0 and all but finitely many n, there exists a character $\chi \in P(n, \chi_0)$ such that $L(\pi, \chi, \frac{1}{2}) \neq 0$. Moreover, Vatsal showed in [6] that if χ_0 has order prime to p and the Hecke field of g is linearly disjoint from the field generated over $\mathbb{Q}$ by the p-th roots of unity, then $L(\pi, \chi, \frac{1}{2}) \neq 0$ for all $\chi \in P(n, \chi_0)$ with n sufficiently large. We remark that this statement differs slightly from the statement given in [6] since the condition on the Hecke field of g was overlooked there. In [7], Vatsal extended the results and methods of [6] to study the algebraic part of the special value $L(\pi, \chi, \frac{1}{2})$ modulo a given prime $\lambda \in \bar{\mathbb{Q}}$ of characteristic l. Vatsal proved

that for all $n \gg 0$, there exists $\chi \in P(n, \chi_0)$ such that

$$ord_\lambda \left(\frac{(L(\pi, \chi, \frac{1}{2})}{\Omega_g^{\mathrm{can}} \mathrm{C}_{\mathrm{csp}}} \right) < ord_\lambda(\mathrm{C}_{\mathrm{Eis}}^2), \tag{2}$$

where $\mathrm{C}_{\mathrm{Eis}}$ is a constant that measures the congruence between g and the space of Eisenstein Series, and $\mathrm{C}_{\mathrm{csp}}$ is a constant that measures the congruence between g and some cusp forms of lower levels. Moreover, if χ_0 has order prime to p, the Hecke field of g is linearly disjoint from the field generated over $\mathbb{Q}$ by the *p-th* roots of unity, and l satisfies certain conditions (see next paragraph), then (2) is true for all $\chi \in P(n, \chi_0)$ with n sufficiently large. We remark that (2) is mistakenly given as an equality in[7], the source of the mistake being an error made in the proof of Proposition 5.3 part (2). We provide a correct version of this result in Proposition 4.12 below. We mention here that establishing the correct statement involves modifying the choice of the constant C_{Eis} as in Definition 4.8 below.

Given $x = [g] \in CM_H(\mathcal{P}^n)$ and a ring class character χ of conductor $\mathcal{P}^n$, we define the Gross-Zagier sum

$$\mathrm{a}(x, \chi) = \frac{1}{|G(n)|} \sum_{\sigma \in G(n)} \chi(\sigma)\psi(\sigma.x).$$

In order to prove that a family of values is non-vanishing, it is a standard technique to compute their average. Hence, in order to show that $\mathrm{a}(x, \chi) \neq 0$ is non-vanishing for some $\chi \in P(n, \chi_0)$, it suffices to show that

$$\mathrm{b}(x, \chi_0) = \sum_{\chi \in P(n, \chi_0)} \mathrm{a}(x, \chi) \neq 0.$$

Vatsal observed in [6] that if the class number of K is prime to $\mathcal{P}$, then all the characters in $P(n, \chi_0)$ are in fact conjugates under the action of $\mathrm{Aut}(\mathbb{C})$. It follows that the sums $\mathrm{a}(x, \chi)$ are also conjugates for all $\chi \in P(n, \chi_0)$. Hence, the non-vanishing of $\mathrm{a}(x, \chi)$ for some $\chi \in P(n, \chi_0)$ forces the non-vanishing of $\mathrm{a}(x, \chi)$ for all $\chi \in P(n, \chi_0)$. In addition, Vatsal noticed in [7] that if l splits completely in the field $\mathbb{Q}(\chi_0)$ generated by the values of χ_0, and if it is inert in the field $\mathbb{Q}(\mu_{p^\infty})$ generated by all p-power roots of unity, then all the characters in $P(n, \chi_0)$ are conjugates under the action of a decomposition group D_λ. Thus, the sums $\mathrm{a}(x, \chi)$ have the same λ-adic valuation for all $\chi \in P(n, \chi_0)$.

Our goal is to prove an analogue of Theorem 1.2 in [7] (see also Proposition 4.1 and Corollary 4.2) for a Hilbert modular form g over a totally real field F, while removing the above mentioned assumptions on l and the class number of K. More precisely, given a character χ_0 of G_0, we prove that

$$ord_\lambda(\mathrm{a}(x, \chi)) < \mu$$

for all $\chi \in P(n, \chi_0)$ with $n \gg 0$, where μ is a constant to be specified at a later stage (see Definition 4.8).

We know by Lemma 2.8 in [3] that we can identify G_0 with its image $G_0(n)$ in $G(n)$ whenever n is sufficiently large. We denote the quotient group $G(n)/G_0(n)$ by $H(n)$. Suppose

that χ_0 is a fixed character of G_0 and let $\chi \in P(n, \chi_0)$. One can express χ as $\chi = \chi_0'\chi_1$, where χ_0' is some character of $G(n)$ inducing χ_0 on $G_0(n) \simeq G_0$, and χ_1 is some character of $H(n)$.

Recall that E_l is an l-adically complete discrete valuation ring containing the Fourier coefficients of g. Enlarge E_l if necessary to contain the values of χ_0 and the p-th roots of unity, and let $E_l(\chi_1)$ be the field obtained by adjoining to E_l the values of χ_1.

Consider the trace of $\mathrm{a}(x, \chi)$ taken from $E_l(\chi_1)$ to E_l:

$$\mathrm{Tr}(\mathrm{a}(x, \chi)) = \sum_{\sigma \in \mathrm{Gal}(E_l(\chi_1)/E_l)} \sigma(\mathrm{a}(x, \chi)).$$

This trace expression is different than the average expression

$$\mathrm{b}(x, \chi) = \sum_{\chi \in P(n, \chi)} \mathrm{a}(x, \chi)$$

considered in the work of Vatsal and Cornut-Vatsal. Nevertheless, the same approach is followed to study both expressions. Evidently, given any $\chi \in P(n, \chi_0)$, the non-vanishing of $\mathrm{Tr}(\mathrm{a}(x, \chi))$ would then imply the non-vanishing of $\mathrm{a}(x, \chi)$.

Notice that

$$\begin{aligned}
\mathrm{Tr}(\mathrm{a}(x, \chi)) &= \frac{1}{|G(n)|} \mathrm{Tr}_{E_l(\chi_1)/E_l} \sum_{\sigma \in G_0(n)} \sum_{\tau \in H(n)} \chi_0'(\sigma)\chi_1(\tau)\psi(\sigma\tau.x) \\
&= \frac{1}{|G(n)|} \sum_{\sigma \in G_0} \chi_0(\sigma) \sum_{\tau \in H(n)} \psi(\sigma\tau.x) \mathrm{Tr}_{E_l(\chi_1)/E_l} \chi_1(\tau) \\
&= \frac{[E_l(\chi_1) : E_l]}{|G(n)|} \sum_{\sigma \in G_0} \chi_0(\sigma) \sum_{\substack{\tau \in H(n) \\ \chi_1(\tau) \in E_l}} \psi(\sigma\tau.x)\chi_1(\tau).
\end{aligned}$$

Let m be the highest power of p dividing the order of χ_0, and put

$$Z(n, m) = \{\tau \in H(n) : \mathrm{order}(\tau) \mid p^m\}.$$

Since $Z(n, m) = \{\tau \in H(n) : \chi_1(\tau) \in E_l\}$, we get

$$\mathrm{Tr}(\mathrm{a}(x, \chi)) = \frac{[E_l(\chi_1) : E_l]}{|G(n)|} \sum_{\sigma \in G_0} \chi_0(\sigma) \sum_{\tau \in Z(n,m)} \chi_1(\tau)\psi(\sigma\tau.x).$$

Lemma 4.1. *For $n \gg 0$, $Z(n, m) \simeq O_F/\mathcal{P}^m$.*

Proof. It follows from the definition of $Z(n, m)$ that $Z(n, m) = \ker(H(n) \twoheadrightarrow H(n - m))$. We also have an isomorphism between $Z(n, m)$ and $\ker(G(n) \twoheadrightarrow G(n - m))$ induced by the natural quotient map $G(n) \twoheadrightarrow H(n)$. The reciprocity map induces an isomorphism between $\ker(G(n) \twoheadrightarrow G(n - m))$ and

$$K^*\hat{O}^*_{\mathcal{P}^{n-m}}/K^*\hat{O}^*_{\mathcal{P}^n} \simeq O^*_{\mathcal{P}^{n-m},\mathcal{P}}/O^*_{\mathcal{P}^{n-m}}O^*_{\mathcal{P}^n,\mathcal{P}}.$$

Notice that $O^*_{\mathcal{P}^{n-m}} = O^*_F$ is contained in $O^*_{\mathcal{P}^n,\mathcal{P}}$ for sufficiently large n, so that

$$\ker(G(n) \twoheadrightarrow G(n-m)) \simeq O^*_{\mathcal{P}^{n-m},\mathcal{P}}/O^*_{\mathcal{P}^n,\mathcal{P}}.$$

On the other hand,

$$O^*_{\mathcal{P}^{n-m},\mathcal{P}}/O^*_{\mathcal{P}^n,\mathcal{P}} = \{1 + a\alpha_{\mathcal{P}}\pi_{\mathcal{P}}^{n-m} \mod O^*_{\mathcal{P}^n,\mathcal{P}} : a \in O_{F,\mathcal{P}}/\mathcal{P}^m\},$$

thus yielding the desired isomorphism. $\square$

Define a function θ_m on $G(\mathbb{A}_f)$ by:

$$\theta_m(g) = \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(g.(1,1,...,\underbrace{\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}}_{\mathcal{P}^{\text{th}}\text{place}},...,1,1)),$$

where $\tau_a \in Z(n,m)$ is the element associated to $a \in O_{F,\mathcal{P}}/\mathcal{P}^m$.

To simplify notation, we identify $(1,1,..,b_{\mathcal{P}},..,1,1) \in G(\mathbb{A}_f)$ with $b_{\mathcal{P}} \in GL_2(F_{\mathcal{P}}) \simeq B^*_{\mathcal{P}}$. For example, we write

$$\theta_m(g) = \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(g.\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}).$$

Lemma 4.2. *The function θ_m has level $H_m = \widehat{R_m}^*$ for some O_F-order R_m which agrees with R outside $\mathcal{P}$. At $\mathcal{P}$, the subgroup $R^*_{m,\mathcal{P}}$ is isomorphic to the Iwahori subgroup*

$$U_1(O_{F,\mathcal{P}}, \mathcal{P}^{\max(2m,\delta)}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(O_{F,\mathcal{P}}, \mathcal{P}^{\max(2m,\delta)}) : a \equiv 1 \mod \pi_{\mathcal{P}}^{\max(2m,\delta)} \right\}.$$

Proof. Let γ be any element in $U_0(O_{F,\mathcal{P}}, \mathcal{P}^{\max(2m,\delta)})$. The Iwahori factorization of γ yields $\gamma = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. We view γ as an element in $G(\mathbb{A}_f)$. For $g \in G(\mathbb{A}_f)$, we have

$$\begin{aligned}
\theta_m(g\gamma) &= \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(g\gamma\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}) \\
&= \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_{ad_1^{-1}d_2})\theta(g\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 - la\pi_{\mathcal{P}}^{-m} & -la^2\pi_{\mathcal{P}}^{-2m} \\ l & 1 + la\pi_{\mathcal{P}}^{-m} \end{pmatrix}) \\
&= \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_{ad_1^{-1}d_2})\theta(g\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix})
\end{aligned}$$

If we further assume that $\gamma \in U_1(O_{F,\mathcal{P}}, \mathcal{P}^{\max(2m,\delta)})$, then $d_1 \equiv d_2 \equiv 1 \mod \pi_{\mathcal{P}}^{\max(2m,\delta)}$. Hence, $\theta_m(g\gamma) = \theta_m(g)$. $\square$

We now make the following important observation. Since $\mathrm{a}(\gamma.x,\chi)=\chi^{-1}(\gamma)\mathrm{a}(x,\chi)$ for any $\gamma\in G(n)$, and $G(n)$ acts simply and transitively on $CM_H(\mathcal{P}^n)$, it suffices to study the λ-adic valuation of $\mathrm{a}(y,\chi)$ for any $y\in CM_H(\mathcal{P}^n)$. Henceforth, we shall take for x the CM point obtained in Lemma 2.2 and fix the choice of representative g determined in the lemma as well. We denote by x_m the class of g in CM_{H_m}.

Lemma 4.3. *Let ψ_m denote the function induced by θ_m on CM_{H_m}. We have*

$$\sum_{\tau\in Z(n,m)}\chi_1(\tau)\psi(\tau.x)=\psi_m(x_m).$$

Proof. By Lemma 4.1, we view $\tau\in Z(m,n)$ as the class of $1+a\alpha_{\mathcal{P}}\pi_{\mathcal{P}}^{n-m}$ in $O^*_{\mathcal{P}^{n-m},\mathcal{P}}/O^*_{\mathcal{P}^n,\mathcal{P}}$ for some $a\in O_{F,\mathcal{P}}/\mathcal{P}^m$. We then identify τ with its image in $GL_2(O_{F,\mathcal{P}})$:

$$\tau\mapsto\begin{pmatrix}1+a\pi_{\mathcal{P}}^{n-m}\mathrm{Tr}\alpha_{\mathcal{P}} & a\pi_{\mathcal{P}}^{n-m}\mathrm{N}\alpha_{\mathcal{P}}\\ -a\pi_{\mathcal{P}}^{n-m} & 1\end{pmatrix}.$$

Write $x=[g]$, where $g\in G(\mathbb{A}_f)$ is as defined in Lemma 2.2. We have

$$\begin{aligned}\sum_{\tau\in Z(n,m)}\chi_1(\tau)\psi(\tau.x)&=\sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m}\chi_1(\tau_a)\psi(\tau_a.x)\\&=\sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m}\chi_1(\tau_a)\theta(\begin{pmatrix}1+a\pi_{\mathcal{P}}^{n-m}\mathrm{Tr}\alpha_{\mathcal{P}} & a\pi_{\mathcal{P}}^{n-m}\mathrm{N}\alpha_{\mathcal{P}}\\ -a\pi_{\mathcal{P}}^{n-m} & 1\end{pmatrix}g)\\&=\sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m}\chi_1(\tau_a)\theta(g\begin{pmatrix}1+a\pi_{\mathcal{P}}^{n-m}\mathrm{Tr}\alpha_{\mathcal{P}} & a\pi_{\mathcal{P}}^{-m}\\ -a\pi_{\mathcal{P}}^{2n-m}\mathrm{N}\alpha_{\mathcal{P}} & 1\end{pmatrix})\\&=\sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m}\chi_1(\tau_a)\theta(g\begin{pmatrix}1 & a\pi_{\mathcal{P}}^{-m}\\ 0 & 1\end{pmatrix})\\&=\psi_m(x_m).\end{aligned}$$

The fourth line follows from the fact that

$$\begin{pmatrix}1+a\pi_{\mathcal{P}}^{n-m}\mathrm{Tr}\alpha_{\mathcal{P}} & a\pi_{\mathcal{P}}^{-m}\\ -a\pi_{\mathcal{P}}^{2n-m}\mathrm{N}\alpha_{\mathcal{P}} & 1\end{pmatrix}=\begin{pmatrix}1 & a\pi_{\mathcal{P}}^{-m}\\ 0 & 1\end{pmatrix}\begin{pmatrix}d_1 & 0\\ 0 & d_2\end{pmatrix}\begin{pmatrix}1 & 0\\ l & 1\end{pmatrix},$$

where $d_1,d_2\in O_{F,\mathcal{P}}$, and $l\in\mathcal{P}^\delta$ for $n\gg 0$. $\square$

Since $\chi_0=1$ on G_2, we have:

$$\mathrm{Tr}(\mathrm{a}(x,\chi))=\frac{|G_2|[E_l(\chi_1):E_l]}{|G(n)|}\sum_{\sigma\in G_0/G_2}\chi_0(\sigma)\psi_m(\sigma.x_m).$$

We can reduce the above sum into something even simpler by means of another level raising step.

Proposition 4.4. *There exists an O_F-order $R_{m,\mathcal{D}}$, a nonzero function $\theta_{m,\mathcal{D}}$ of level $H_{m,\mathcal{D}} = \widehat{R_{m,\mathcal{D}}}^*$ on $G(\mathbb{A}_f)$, and for each $n \geq 0$, a Galois equivariant map $x \mapsto x_{m,\mathcal{D}}$ from $CM_H(\mathcal{P}^n)$ to $CM_{H_{m,\mathcal{D}}}(\mathcal{P}^n)$ such that*

$$\psi_{m,\mathcal{D}}(x_{m,\mathcal{D}}) = \sum_{\tau \in G_1/G_2} \chi_0(\tau)\psi_m(\tau.x_m),$$

where $\psi_{m,\mathcal{D}} = \theta_{m,\mathcal{D}} \circ \mathrm{red}$.

Proof. The reader is referred to the proof of Lemma 5.9 in [3] □

Hence, the trace expression simplifies to

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \frac{|G_2|[E_l(\chi_1) : E_l]}{|G(n)|} \sum_{\sigma \in G_0/G_1} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.x_{m,\mathcal{D}}).$$

We now study the λ-adic valuation of the sum

$$\sum_{\sigma \in \mathcal{R}} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.x_{m,\mathcal{D}}),$$

where $\mathcal{R}$ is a set of representatives for G_0/G_1 containing 1.

Definition 4.5. *Let k be any ring. A k-valued function ϕ on M_H is said to be Eisenstein if it factors through N_H via the map* c, *where as ϕ is said to be exceptional if there exists $z \in \mathrm{N}_H$ such that ϕ is constant on $c^{-1}(\sigma.z)$ for all $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$.*

Choose an ideal $\mathcal{C}$ in O_F such that $\mathrm{nrd}(H)$ contains all elements of $\widehat{O_F}^*$ congruent to 1 modulo $\mathcal{C}$. Such an integral ideal exists because $\mathrm{nrd}(H)$ is open in $\widehat{F}^*$. For a finite prime v of F, let q_v be the cardinality of the residue class field at v. Denote by S the set of all finite places of F that do not divide $\mathcal{N}$ and correspond to a principal prime ideal aO_F with $a \equiv 1 \mod C$ and a is totally positive.

Lemma 4.6. *If ϕ is Eisenstein modulo λ^r for some positive integer r, then $a_v \equiv q_v + 1 \mod \lambda^r$ for all $v \in S$.*

Proof. Let v be a finite place in F corresponding to a principal prime ideal $Q = aO_F$ with $a \equiv 1 \mod C$ and a is totally positive. Choose $x \in \mathrm{M}_H$ such that $\phi(x)$ is a λ-adic unit. By definition, we know that

$$T_v\phi(x) = \sum_{i \in I_v} \phi(x\eta_{v,i}).$$

Here $H_v \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} H_v = \coprod_{i \in I_v} \eta_{v,i} H_v$. Notice that $c(x\eta_{v,i}) = c(x\eta_v)$, where

$$\eta_v = (1, .., 1, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1, ..., 1).$$

Since ϕ is Eisenstein modulo λ^r, we get

$$T_v\phi(x) \equiv (1+q_v)\phi(x\eta_v) \mod \lambda^r.$$

Choose $d \in G(\mathbb{Q})$ such that $\mathrm{nrd}(d) = a$. Notice that $\mathrm{nrd}(\eta_v^{-1}d) = (a, ..., a, 1, a, ..., a) \equiv 1 \mod C$. We thus obtain an element $h \in H$ such that $\mathrm{nrd}(h) = \mathrm{nrd}(\eta_v^{-1}d)$. Hence,

$$\phi(x\eta_v) \equiv \phi(x\eta_v d^{-1}) \equiv \phi(xh^{-1}) \equiv \phi(x) \mod \lambda^r.$$

On the other hand, we know that $T_v\phi(x) = a_v\phi(x)$. Putting all of this together gives $a_v\phi(x) \equiv (1+q_v)\phi(x) \mod \lambda^r$, which implies that $a_v \equiv 1+q_v \mod \lambda^r$ since $\phi(x)$ is a λ-adic unit. □

Lemma 4.7. *If ϕ is exceptional but non-Eisenstein modulo λ^r for some positive integer r, then a_v is a λ-adic non-unit for all finite places v of F that are inert in K and do not divide $\mathcal{N}$.*

Proof. The argument given here is drawn from [3]. Recall that we have an action of the group $\mathrm{Gal}_K^{\mathrm{ab}}$ on N_H, and one can show that there are at most two $\mathrm{Gal}_K^{\mathrm{ab}}$-orbit in N_H. If there were only one $\mathrm{Gal}_K^{\mathrm{ab}}$-orbit in N_H, then any exceptional function on M_H would also be Eisenstein. Since ϕ is exceptional and non-Eisenstein modulo λ^r, we know there must be exactly two $\mathrm{Gal}_K^{\mathrm{ab}}$-orbits in N_H, which we denote by X and Y with ϕ being constant modulo λ^r on $\mathrm{c}^{-1}(z)$ for all $z \in X$. Since ϕ is non-Eisenstein modulo λ^r, there exist $y \in Y$ and some $x_1, x_2 \in \mathrm{c}^{-1}(y)$ such that $\phi(x_1) \not\equiv \phi(x_2) \mod \lambda^r$.
Let v be a finite place of F that is inert in K and does not divide $\mathcal{N}$. For any $x \in \mathrm{M}_H$, we know that

$$T_v\phi(x) = a_v\phi(x) = \sum_{i \in I_v} \phi(x\eta_{v,i}).$$

We also know that if $x \in \mathrm{c}^{-1}(y)$ then $x\eta_{v,i} \in \mathrm{c}^{-1}(\mathrm{Frob}_v.y)$. Since v is inert in K, we get $\mathrm{Frob}_v.y \in X$, so that ϕ is constant modulo λ^r on $\mathrm{c}^{-1}(\mathrm{Frob}_v.y)$ with $\phi(v, y)$ being the common value. Hence,

$$a_v\phi(x_1) \equiv (1+q_v)\phi(v, y) \equiv a_v\phi(x_2) \mod \lambda^r.$$

It follows that a_v is a λ-adic non-unit, since otherwise $\phi(x_1)$ and $\phi(x_2)$ would be congruent modulo λ^r. □

We shall assume henceforth that g satisfies the condition: a_v is a λ-adic unit for some v inert in K, $v \nmid \mathcal{N}$.

Definition 4.8. *Let μ be the smallest integer such that $a_v \not\equiv 1 + q_v \mod \lambda^\mu$ for some $v \in S, v \nmid \mathcal{D}$.*

It follows immediately from the definition of μ that the function θ is non-exceptional modulo λ^μ.

We let f be the inertia degree of $\mathcal{P}$ over p. If λ lies above p ($l = p$), we let e be the corresponding ramification index. In this case, we denote by k the ring $E_l/\lambda^{emf+\mu}E_l$. If λ does not lie above p ($l \neq p$), we denote by k the ring $E_l/\lambda^\mu E_l$. We shall view θ, θ_m and $\theta_{m,\mathcal{D}}$ as k-valued functions.

Proposition 4.9. *The function $\theta_m : M_{H_m} \to k$ is a non-zero eigenfunction for all Hecke operators T_v $(v \nmid \mathcal{P}\mathcal{N}'\mathcal{D}')$ with $T_v\theta_m = a_v\theta_m$.*

Proof. It is clear that θ_m is an eigenfunction for all Hecke operators T_v $(v \nmid \mathcal{P}\mathcal{N}'\mathcal{D}')$ with $T_v\theta_m = a_v\theta_m$. If $\theta_m = 0$ as a k-valued function, then

$$\begin{aligned}
0 &= \sum_{u \in (O_{F,\mathcal{P}}/\mathcal{P}^m)^*} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} .\theta_m \\
&= \sum_{u \in (O_{F,\mathcal{P}}/\mathcal{P}^m)^*} \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_{ua}) \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix} .\theta \\
&= \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix} .\theta \sum_{u \in (O_{F,\mathcal{P}}/\mathcal{P}^m)^*} \chi_1(\tau_{ua})
\end{aligned}$$

Let $q = p^f$ denote the cardinality of the residue class field $O_F/\mathcal{P}$. By means of Lemma 4.11 below, we get

$$\begin{aligned}
0 &= q^{m-1}(q-1)\theta - q^{m-1} \sum_{a \in (O_{F,\mathcal{P}}/\mathcal{P})^*} \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-1} \\ 0 & 1 \end{pmatrix} .\theta \\
&= q^m\theta - q^{m-1} \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}} \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-1} \\ 0 & 1 \end{pmatrix} .\theta \\
&= q^{m-1} \left(q\theta - \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}} \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-1} \\ 0 & 1 \end{pmatrix} .\theta \right) \\
&= q^{m-1}\theta^+.
\end{aligned}$$

This yields a contradiction since $q^{m-1}\theta^+$ is non-zero by Lemma 4.12 in [3]; the proof of this lemma uses the fact that θ is non-eisenstein modulo λ^μ. The reader is referred to [3] for a description of the function θ^+ and its properties (see, for example, Section 1.6, Theorem 5.10 and the Appendix). ∎

Corollary 4.10. *θ_m is non-exceptional as a k-valued function.*

Lemma 4.11. *For $a \in O_F/\mathcal{P}^m$, we have*

$$\sum_{u \in (O_F/\mathcal{P}^m)^*} \chi_1(\tau_{ua}) = \begin{cases} q^{m-1}(q-1) & a \in \mathcal{P}^m \\ -q^{m-1} & a \in \mathcal{P}^{m-1}/\mathcal{P}^m \text{ and } a \notin \mathcal{P}^m \\ 0 & \text{otherwise} \end{cases}$$

Proof. The statement of the lemma follows trivially for $a \equiv 0 \mod \mathcal{P}^m$. For the remaining cases, we write

$$\sum_{u \in (O_F/\mathcal{P}^m)^*} \chi_1(\tau_{ua}) = \sum_{u \in O_F/\mathcal{P}^m} \chi_1(\tau_{ua}) - \sum_{u \in \mathcal{P}/\mathcal{P}^m} \chi_1(\tau_{ua})$$

Notice that if $a \in \mathcal{P}^{m-1}/\mathcal{P}^m$, we have

$$\sum_{u \in O_F/\mathcal{P}^m} \chi_1(\tau_{ua}) = 0 \text{ and } \sum_{u \in \mathcal{P}/\mathcal{P}^m} \chi_1(\tau_{ua}) = q^{m-1}.$$

Otherwise, we get

$$\sum_{u \in O_F/\mathcal{P}^m} \chi_1(\tau_{ua}) = \sum_{u \in \mathcal{P}/\mathcal{P}^m} \chi_1(\tau_{ua}) = 0$$

. □

Proposition 4.12. *The function $\theta_{m,\mathcal{D}} : M_{H_{m,\mathcal{D}}} \to k$ is a non-zero eigenfunction for all Hecke operators T_v away from $\mathcal{P}\mathcal{N}'\mathcal{D}'$ with $T_v\theta_{m,\mathcal{D}} = a_v\theta_{m,\mathcal{D}}$.*

Proof. By definition (see [3] p. 57),

$$\theta_{m,\mathcal{D}} = \sum_{d|\mathcal{D}'} \chi_0(\sigma_d)(\alpha_d.\theta_m),$$

where $\alpha_d = \prod_{Q|d} \alpha_Q$, and α_Q is any element in $R_Q \sim M_2(O_{F,Q})$. Notice that $\theta_{m,\mathcal{D}}$ is left-invariant under $H_{m,\mathcal{D}} = \widehat{R}_{m,\mathcal{D}}$ where $R_{m,\mathcal{D}}$ is the unique O_F-order which agrees with R_m outside $\mathcal{D}'$ and equals $R_Q \cap \alpha_Q R_Q \alpha_Q^{-1}$ at $Q \mid \mathcal{D}'$.

If we fix a prime divisor Q of $\mathcal{D}'$, it is easy to see that $\theta_{m,\mathcal{D}}$ can be rewritten as

$$\theta_{m,\mathcal{D}} = \sum_{d|\frac{\mathcal{D}'}{Q}} \chi_0(\sigma_d)(\alpha_d.\theta_m) + \chi_0(\sigma_Q)\alpha_Q. \sum_{d|\frac{\mathcal{D}'}{Q}} \chi_0(\sigma_d)(\alpha_d.\theta_m).$$

Let ϑ_1 and ϑ_2 be k-valued functions on M_{H_m} satisfying $T_v\vartheta_i = a_v\vartheta_i$ for all $v \nmid \mathcal{P}\mathcal{N}'\mathcal{D}'$. We claim that any nontrivial linear combination $a\vartheta_1 + b\alpha_Q.\vartheta_2$ is non-zero in k. If $a\vartheta_1 + b\alpha_Q.\vartheta_2 = 0$ for some scalars a and b, then $a\vartheta_1 = -b\alpha_q.\vartheta_2$ is fixed under the group spanned by R_Q^* and $\alpha_Q R_Q^* \alpha_Q^{-1}$ which contains the kernel of the reduced norm map $B_\mathcal{P} \to F_\mathcal{P}$. It follows from the strong approximation theorem ([8] p. 81) that ϑ_1 factors through the norm map as a k-valued function, which is a contradiction to the fact that $a_v \not\equiv q_v + 1 \mod \lambda^\mu$ for some $v \in S$ (Lemma 4.6). Hence, $a\vartheta_1 + b\alpha_Q.\vartheta_2$ is non-zero. Not only this, but $a\vartheta_1 + b\alpha_Q.\vartheta_2$ is also an eigenfunction for all T_v ($v \nmid \mathcal{P}\mathcal{N}'\mathcal{D}'$) with the same eigenvalues as ϑ_1 and ϑ_2.

In light of the above observation, we proceed by induction on the number of prime ideal divisors of $\mathcal{D}'$ to prove that $\theta_{m,\mathcal{D}}$ is non-zero and satisfies $T_v\theta_{m,\mathcal{D}} = a_v\theta_{m,\mathcal{D}}$ for all $v \nmid \mathcal{P}\mathcal{N}'\mathcal{D}'$. This reduces the problem to the case of θ_m which satisfies the required hypothesis by Proposition 4.9. □

Corollary 4.13. *$\theta_{m,\mathcal{D}}$ is non-exceptional as a k-valued function.*

Now we state and prove the main result in this paper. This result gives an upper bound for the l-adic valuation of the sum

$$\sum_{\tau \in \mathcal{R}} \chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.x_{m,\mathcal{D}}),$$

which we recall is related to the Gross-Zagier sum $\mathrm{a}(x,\chi)$ by the formula

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \frac{|G_2|[E_l(\chi_1):E_l]}{|G(n)|}\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.x_{m,\mathcal{D}}).$$

Theorem 4.14. *Let χ_0 be any character of G_0. For any $x \in CM_{H_{m,\mathcal{D}}}(\mathcal{P}^n)$ with $n \gg 0$, there exists some $y \in G(\infty).x$ such that*

$$\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y) \not\equiv 0 \quad (\textit{in } k).$$

Proof. We follow the proof of Corollary 5.7 in [3]. Since $\theta_{m,\mathcal{D}}$ is non-exceptional as a k-valued function, there exists $\sigma \in G(\infty)$ such that $\theta_{m,\mathcal{D}}$ is non-constant as a k-valued function on $\mathrm{c}^{-1}(\mathrm{c}\circ\mathrm{red}(\sigma.x))$. Choose $p_1, p_2 \in \mathrm{c}^{-1}(\mathrm{c}\circ\mathrm{red}(\sigma.x))$ such that $\theta_{m,\mathcal{D}}(p_1) \not\equiv \theta_{m,\mathcal{D}}(p_2)$ (in k). If n is sufficiently large, Theorem 3.1 guarantees the existence of $y_1, y_2 \in G(\infty).x$ such that

$$\mathrm{red}(y_1) = p_1, \quad \mathrm{red}(y_2) = p_2$$

and

$$\mathrm{red}(\tau.y_1) = \mathrm{red}(\tau.x) = \mathrm{red}(\tau.y_2) \quad \text{ for all } \tau \neq 1 \text{ in } \mathcal{R}.$$

We thus obtain

$$\begin{aligned}\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y_1) - \sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y_2) &= \theta_{m,\mathcal{D}}(p_1) - \theta_{m,\mathcal{D}}(p_2)\\ &\not\equiv 0 \text{ (in } k).\end{aligned}$$

Therefore, at least one of the sums $\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y_1)$ or $\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y_2)$ is non-zero in k. ☐

Finally, we remark that one can easily obtain a lower bound on the l-adic valuation of the Gross-Zagier sum $\mathrm{a}(x,\chi)$. In fact, let ν be the largest integer such that θ is Eisenstein modulo λ^ν. Then

$$\begin{aligned}\sum_{\sigma\in G(n)}\chi(\sigma)\theta\circ\mathrm{red}(\sigma.x) &= \sum_{\sigma\in G(n)}\chi(\sigma)\theta(\mathrm{red}(\sigma.x))\\ &\equiv \sum_{\sigma\in G(n)}\chi(\sigma)\theta(\mathrm{c}\circ\mathrm{red}(\sigma.x))\\ &\equiv \sum_{\sigma\in G(n)}\chi(\sigma)\theta(\sigma.\mathrm{c}\circ\mathrm{red}(x))\\ &\equiv \sum_{\sigma\in G(n)}\chi(\sigma)\theta(\mathrm{nrd}(\beta)\mathrm{c}\circ\mathrm{red}(x))\\ &\equiv 0 \mod \lambda^\nu,\end{aligned}$$

where the last line follows from the orthogonality property of group characters. Hence,

$$ord_\lambda\left(\sum_{\sigma\in G(n)}\chi(\sigma)\psi(\sigma.x)\right)\geq\nu.$$

It is obvious that this simple observation combined with Theorem 4.14 would give an exact value for the l-adic valuation of $\mathrm{a}(x,\chi)$ if we have $\nu+1=\mu$. However, it is not clear to us whether the statement $\nu+1=\mu$ is true or not. This is a very interesting question, but we choose not to discuss it in this work. We remark only that the answer seems to be connected to multiplicity-one-type results for the component group of a Shimura curve at Eisenstein primes.

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